Number Theory (Problem Set)

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1. Integers a, b, c satisfy $a+b-c=1$, $a^2+b^2-c^2=-1$. Find the sum of all possible distinct values of $a^2+b^2+c^2$.

2. Let a and b be natural numbers such that $a+b, a-2b, 2a-b$ are all distinct squares. What is the smallest possible value of b?

3. Find the least possible value of $a+b$, where $a, b \in \mathbb{Z}^+$ such that $11 \mid a+13b$ and $13 \mid a+11b$.

4. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n))$ = number of divisors of n. Prove that if p is a prime, then $f(p)$ is a prime.

5. $p_1 = 2, p_2 = 3, p_3 =$ largest prime factor of $(p_1p_2+1) = 7, p_4 =$ largest prime factor of $(p_1p_2p_3+1) = 43$. In general, $p_n =$ largest prime factor of $(p_1p_2 \dots p_{n-1}+1)$. Prove $p_n \neq 5\ \forall n \in \mathbb{N}$.

6. Show that the quadratic equation $x^2+7x-14(q^2+1)=0$ (with $q \in \mathbb{Z}$) has no integral root.

7. $N = 13 \times 17 \times 41 \times 829 \times 56659712633$ is an 18-digit number in which 9 of the 10 digits (0 to 9) each appear exactly twice. Find the sum of the digits of N. [Hint: $n -$ sum of digits of n is divisible by 9.]

8. $n \in \mathbb{N}$ has k divisors $d_1 < d_2 < \cdots < d_k$, and $d_5^2 + d_6^2 = 2n+1$. Find k and n.

9. Solve $y^2 = x^3 + 7$ in integers.

10. For all $n \in \mathbb{N}$, define $s(n) = \#\{x, y \in \mathbb{Z}^+ \mid \frac{1}{x} + \frac{1}{y} = \frac{1}{n}\}$. Determine the set of n such that $s(n) = 5$.

11. Prove that $2^n \nmid n!$ for all $n \in \mathbb{N}$.

12. Determine the last three digits of $\sum_{n=2}^{10000000} (n^7 + n^5)$.

13. Given $g(x) = (4a-3d)x^5 + (4b-3e)x^4 + (4c-3f)x^3 + (4d-3a)x^2 + (4e-3b)x + (4f-3c)$, where $a, b, c, d, e, f \in \{1, 2, \dots, 9\}$ and $g(10) = 0$, find $a+b+c+d+e+f$. [Hint: Think in terms of number representation.]

14. Solve $2^m + 3^n = a^2$, for natural numbers m, n, a.

15. Find all possible values of $a+b+c+d$ where $ab \cdot cd = ddd$, and a, b, c, d are distinct digits.

16. Prove that there exist 100 consecutive natural numbers with exactly 3 primes.

17. In the sequence $\{101, 10101, 1010101, \dots\}$, how many terms are prime?

18. Give all possible representations of 2022 as a sum of at least two consecutive positive integers and prove that these are the only representations.

19. Prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = n+1$. Here $\mathbb{N}$ is the positive integers $\{1, 2, 3, \dots\}$.

20. Find all positive integers a, b, c, d, and n satisfying $n^a + n^b + n^c = n^d$ and prove that these are the only such solutions.

21. Given $n \in \mathbb{N}$ has k divisors: $1 = d_1 < d_2 < \cdots < d_k = n$, and $d_{13} + d_{14} + d_{15} = n$. Find k.

22. Let $A = \sum_{n=1}^{50} \left(x^n + \frac{1}{x^n}\right)$, $B = \sum_{n=1}^{10} \left(x^{5n-2} + \frac{1}{x^{5n-2}}\right)$, and $x + \frac{1}{x} = 4$. Find $\frac{A}{B}$.

23. Consider the series $50 + n^2$: $51, 54, 59, 66, 75, \ldots$. Taking GCD of consecutive terms gives: $3, 1, 1, 3, \ldots$. What is the sum of all distinct elements in the GCD series?

24. Prove that there exist 100 consecutive natural numbers such that exactly 3 of them are prime. [Hint: Use Discrete IVT]

25. For any positive integer n, define $f(n)$ to be the smallest positive integer that does not divide n. For example, $f(1) = 2$, $f(6) = 4$. Prove that for any positive integer n, either $f(f(n))$ or $f(f(f(n)))$ must be equal to 2.

26. Define the series: $A(1) = 1$; $A(n) = f(m)$ number of $f(m)$'s followed by $f(m)$ number of 0's, where $m =$ number of digits in $A(n-1)$, and $f(m) = m \bmod 9$. Find sum of digits of $A(30)$.

27. If $x \in \mathbb{R}^+$ and $\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor = 88$, find $\lfloor 7x\rfloor$.

28. Find the last non-zero digit of 2026!.

29. The number 258145266804692077858261512663 is the 13th power of a natural number. Find the base number.

30. Let $a_1, a_2, \ldots, a_n \in \mathbb{Z}$. Show that there exists k, r such that $a_k + a_{k+1} + \cdots + a_{k+r}$ is divisible by n.

31. Let $S = \{n \in \mathbb{Z}^+ \mid n \text{ and } n+1 \text{ both have exactly 4 divisors and equal sum of divisors}\}$. Let $m = |S|$ and $b = \sum S$. Find m and b.

32. Let x_n denote the n^{th} non-square positive integer. Then

$$x_1 = 2, \quad x_2 = 3, \quad x_3 = 5, \quad x_4 = 6, \text{ etc.}$$

For a positive real number x, denote the integer closest to it by $\langle x\rangle$. If $x = m + 0.5$, where m is an integer, then $\langle x\rangle = m$. Eg. $\langle 1.2\rangle = 1$, $\langle 2.8\rangle = 3$, $\langle 3.5\rangle = 3$.

Show that:

$$x_n = n + \left\langle\sqrt{n}\right\rangle$$

33. A corona sequence is an increasing sequence of 16 consecutive odd integers with sum a perfect cube. How many such sequences contain only 3-digit numbers?

34. Let $P(x)$ be the remainder when $(x+7)^{100}$ is divided by $x^2 - x - 1$. Find the remainder when $P(x)$ is divided by 11.

35. Let $a_1 < a_2 < \cdots < a_n \in \mathbb{Z}^+$, such that $\frac{1}{a_1} + \cdots + \frac{1}{a_n} = 1$. Give a general method to find such tuples.

36. Let S_n be the sum of reciprocals of all non-empty subsets of $\{1, 2, \ldots, n\}$. Show:
    (A) $S_n = \frac{1}{n} + \left(1 + \frac{1}{n}\right) S_{n-1}$
    (B) Prove using (A) that $S_n = n$
    (C) Prove without using (A) that $S_n = n$.

37. Find all $n \in \mathbb{Z}^+$ such that $5^n + 1 \equiv 0 \pmod 7$.

38. Compute $\left\{\frac{a}{p}\right\} + \left\{\frac{2a}{p}\right\} + \cdots + \left\{\frac{(p-1)a}{p}\right\}$, where p is a prime, $a \in \mathbb{Z}^+$, and $p \nmid a$.

39. Let

$$A = \sum_{i=1}^{n} \left(\left\lfloor\frac{n}{i}\right\rfloor - \left\lfloor\frac{n-1}{i}\right\rfloor\right), \quad B = \sum_{i=1}^{n} \left(\left\lfloor\frac{n}{i}\right\rfloor^2 - \left\lfloor\frac{n-1}{i}\right\rfloor^2\right)$$

Prove that number of divisors of n is A, and sum of divisors is $\frac{A+B}{2}$.

40. Find all 3 digit positive integers abc whose base 9 representation is bca, where a, b, c are not necessarily distinct.

41. Let $d = \gcd(n^2 + 20, (n+1)^2 + 20)$. Show $d \mid 81$.

42. Let $d_n = \gcd(n^3 + n^2 + 1, n^3 + n + 1)$. Prove that $d_n | 3$.

43. Let $p > 5$ be a prime and $\frac{1}{p} = 0.\overline{a_1 a_2 \ldots a_r}$. Prove that $10^r \equiv 1 \pmod{p}$.

44. Is there a natural number n such that $n!$ ends with exactly 2025 zeros?

45. Prove that every positive rational number can be uniquely written as:

$$a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!}$$

where $0 \leq a_i \leq i - 1 \ \forall i > 1$.

46. Let $n \geq 2$ be an integer. Let m be the largest integer which is less than or equal to n, and which is a power of 2. Put $l_n =$ least common multiple of $1, 2, \ldots, n$. Show that $\frac{l_n}{m}$ is odd, and that for every integer $k \leq n$, $k \neq m$, $\frac{l_n}{k}$ is even. Hence, prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is not an integer.

47. For any positive integer n, and $i = 1, 2$, let $f_i(n)$ denote the number of divisors of n of the form $3k + i$ (including 1 and n). Define, for any positive integer n,

$$f(n) = f_1(n) - f_2(n).$$

Find the value of $f(5^{2022}), f(2^{12022})$.

48. Find all the possible two-digit numbers ab such that

$$ab = 4(a! + b!),$$

where ab is the number whose first digit is a and second digit is b.

49. If $\sqrt{\log_b n} = \log_b \sqrt{n}$ and $b \log_b n = \log_b(bn)$, then the value of n is equal to $\frac{j}{k}$, where j and k are relatively prime. What is $j + k$?

50. Find the number of pairs of primes (p, q) for which $p - q$ and $pq - q$ are both perfect squares.